JOURNAL OF DIFFERENTIAL EQUATIONS **131**, 267–276 (1996)
ARTICLE NO. 0164

Analytical Non-integrability of the Truncated Two Fixed Centres Problem in the Symmetric Case

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Received February 14, 1996

The Problem of two fixed centres is an integrable Hamiltonian system. If one truncates the Taylor expansion of the potential of this problem (in the symmetric case) at any order $\geqslant 3$, we prove that one obtains a system which does not admit

Then we use this result to prove that the Vinti Problem, truncated at any order $\geqslant 3$, is analytically non-integrable. © 1996 Academic Press, Inc.

Le Problème des deux centres fixes est un système hamiltonien intégrable. Si l'on tronque à un ordre arbitraire $\geqslant 3$ le développement de Taylor du potentiel de ce problème (dans le cas symétrique), on montre que le système obtenu n'admet aucune intégrale première méromorphe et fonctionnellement indépendante de l'énergie et du moment angulaire. La démonstration est principalement fondée sur le critère de non intégrabilité pour les potentiels homogènes, que Yoshida a déduit du théorème de Ziglin. Nous utilisons ensuite ce résultat pour établir que le Problème de Vinti, tronqué à un ordre arbitraire $\geqslant 3$, est non intégrable analytiquement. © 1996 Academic Press, Inc.

1. INTRODUCTION

For several years, many studies have been devoted to the proof of analytical non-integrability of Hamiltonian systems. These proofs are based on a theorem of Ziglin ([Z], 1983) which gives necessary conditions to be satisfied by an integrable Hamiltonian system ([It], [C-R], [Y_1], [Y_2], [Y-R-G], [F-I], [I-S]). In particular, Yoshida [Y_1] in 1987 constructed a criterion of non-integrability for Hamiltonian systems with two degrees of freedom, the potential of which is a homogeneous function. Then, he considered the Toda Hamiltonian with two degrees of freedom (which is integrable) and he proved that, truncating the Taylor expansion of the potential at any order, one obtains a system having no first integral, meromorphic and independent of the Hamiltonian function [Y_2] [Y-R-G].

0022-0396/96 $18.00

In a similar approach, we start here from the two fixed centres problem, which is an integrable Hamiltonian system [L]. We truncate the potential expansion of this problem, in the symmetric case, at any order $\geqslant 3$ and obtain a Hamiltonian depending on the sum of Legendre polynomials. We prove that such a system is analytically non-integrable, whatever is the order $\geqslant 3$ of truncation.

Then we give an application to the Vinti Problem, in which the gravitational potential of an oblate planet is represented by the attraction of two fixed points separated by an imaginary distance.

2. ZIGLIN'S THEOREM AND YOSHIDA'S CRITERION

We are interested here in Hamiltonian systems with two degrees of freedom and we shall state Ziglin's theorem for this case. It gives necessary conditions to be satisfied by the linearized equations along a family of particular solutions of a Hamiltonian system which admits a second meromorphic integral:

Ziglin's Theorem [Z] [It]

Assume that a Hamiltonian system has a family of particular solutions Γ_h parametrized by periodic functions of the complex time and depending analytically on a real parameter $h \in (h_1, h_2)$. Let G be the monodromy group of the normal variational equation associated to the solution Γ_h. We say that $g \in G$ is non-resonant if every eigenvalue of g is different from a root of unity. If the Hamiltonian system has a meromorphic integral F, functionally independent of H in a neighborhood of Γ_h, and if the monodromy group G contains a non-resonant element g_1, then for any $g_2 \in G$, the commutator $g^* = g_2^{-1} \cdot g_1^{-1} \cdot g_2 \cdot g_1$ satisfies: either $g^* = \mathrm{Id}$ or $g^* = (g_1)^2$.

By this theorem, we have sufficient conditions of non-integrability: if the necessary conditions of Ziglin are not satisfied by a Hamiltonian system, it is not analytically integrable. For instance, this will happen if we can find two non-resonant monodromy matrices g_1 and g_2 which do not cummute.

Using these sufficient conditions, Yoshida [Y_1] gave a criterion of non-integrability for Hamiltonian systems with a homogeneous potential. Suppose that, in the Hamiltonian function

$$H(q_1, q_2, p_1, p_2) = \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{1}$$

V is a homogeneous function of degree $k \in Z^*$.

Then the Hamiltonian system admits a family of particular solutions of the form: $\underline{q}=\underline{c}.\ \Phi(t)$ where $\Phi(t)$ is determined by the first integral of energy:

$$\frac{k}{2}\left(\frac{d\Phi}{dt}\right)^2 = 1-(\Phi(t))^k$$

By analytic continuation, the function $\Phi(t)$ $(t \in C)$ defines a Riemann surface Γ.

Along these solutions Γ, the normal variational equation can be written as following:

$$\frac{d^2\eta}{dt^2}+\lambda_k\cdot(\Phi(t))^{k-2}\cdot\eta=0. \tag{2}$$

Then, to every loop γ on the Riemannian surface Γ, we can associate the symplectic matrix g which characterizes the evolution of the fundamental solutions of equation (2) along γ. This matrix g depends only on the homotopy class of the loop γ. The set of these matrices is the monodromy group G of Eq. (2).

Yoshida's Criterion $[\mathrm{Y}_1]$

Let us define the following set of real numbers, depending on the degree k $(k \leqslant -3)$:

$$S_k =]1;\ +\infty[\ \cup\ ]-|k|+2;0[\ \cup\ ]-3\ |k|+3;\ -|k|-1[\ \cup\cdots$$

$$\cup\left]-\frac{j(j+1)|k|}{2}+j+1;\ -\frac{j(j-1)|k|}{2}-j+1\right[\ \cup\cdots$$

Let λ_k be the coefficient defined by Eq. (2).

If $\lambda_k \in S_k$, then the Hamiltonian system defined by (1) does not admit any analytic integral independent of H.

To prove this criterion, Yoshida remarks that the normal variational Eq. (2) can be transformed, by a change of time, into the Gauss hypergeometric equation. Now the monodromy group of this equation is known explicitly [H] [In] [C-R]: it is generated by two symplectic matrices g_1 and g_2 which can be chosen in such a way that the explicit calculation is possible. Then, if $\lambda_k \in S_k$, it can be proved that: $\operatorname{tr} g_1 > 2$ and $\operatorname{tr} g_2 > 2$. Therefore g_1 and g_2 are non-resonant and do not commute $[\mathrm{Y}_1]$ $[\mathrm{Y}_2]$.

3. EXPANSION OF THE HAMILTONIAN OF THE TWO FIXED CENTRES PROBLEM

Let us consider the potential created in the threee-dimensional space R^3, by two fixed ponctual masses m_1 and m_2 on the z-axis, at distances equal to c_1 and c_2 from the origin. Expressed in the cylindrical coordinates (ρ, θ, z) the potential is:

$$U: R_+ \times R \backslash \{(0, c_1), (0, c_2)\} \to R_+^*$$

$$U(\rho, z) = \frac{1}{r}\left(\frac{m_1}{\left(1 - 2\frac{z}{r}\frac{c_1}{r} + \left(\frac{c_1}{r}\right)^2\right)^{1/2}} + \frac{m_2}{\left(1 - 2\frac{z}{r}\frac{c_2}{r} + \left(\frac{c_2}{r}\right)^2\right)^{1/2}}\right),$$

where $r^2 = \rho^2 + z^2$

The Hamiltonian function of this problem is defined by

$$H(\rho, z, p_\rho, p_\theta, p_z) = \frac{1}{2}\left(p_\rho^2 + \frac{p_\theta^2}{\rho^2} + p_z^2\right) - U(\rho, z).$$

It is well known that this problem is integrable by quadratures ([L] page 93, and [B]). First, this system admits the angular momentum integral: $p_\theta = C$, and then the choice of elliptic coordinates allows to separate the Hamilton–Jacobi equation of the problem and to integrate the differential equations by quadratures.

Now, we shall develop the potential $U(\rho, z)$ in series expansion. If we truncate this expansion at some order, what can be said about the integrability of the correspondent Hamiltonian system? We consider here a situation similar to the problem studied by Yoshida [Y_2] when he proved that the truncated Toda Hamiltonian is not integrable at any order.

From now on, we shall consider only the symmetric problem defined by: $m_1 = m_2 = M/2$ and $c_2 = -c_1 = c > 0$. We have then to develop the function $U(\rho, z)$ with respect to $1/r$, by using the following expansion

$$\frac{1}{(1 - 2\alpha x + x^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(\alpha) \cdot x^n,$$

where $P_n(\alpha)$ is the Legendre polynomial of degree n ([N-U], page 28). This expansion is convergent if $|\alpha| < 1$ and $|x| < 1$.

Then the potential $U(\rho, z)$ can be written in the following form

$$U(\rho, z) = \frac{1}{r}\sum_{p=0}^{\infty}\left(\frac{c}{r}\right)^{2p} \cdot P_{2p}\left(\frac{z}{r}\right).$$

This expansion is convergent in the domain

$$\{(\rho, z) \in R_+^* \times R: \rho^2 + z^2 > c^2\}.$$

As $P_{2p}(z/r)$ is homogeneous of degree 0 with respect to ρ and z, the expansion of $U(\rho, z)$ is the sum of homogeneous terms of degrees -1, -3, -5, ...

4. THE TRUNCATED PROBLEM

Now we consider the expansion of $U(\rho, z)$ truncated to the order $2n+1$ $(n \in N)$.

Because of the first integral: p_θ, the Hamiltonian function H_{2n+1} corresponding to the truncated potential, in restriction to the invariant manifold $p_\theta = C$, is defined by

$$H_{2n+1}(\rho, z, p_\rho, p_z) = \frac{1}{2}(p_\rho^2 + p_z^2) + \frac{C^2}{2\rho^2} - \sum_{p=0}^{n} \frac{c^{2p}}{r^{2p+1}} P_{2p}\left(\frac{z}{r}\right). \tag{3}$$

Remark. For the two fixed centres potential function, $r=0$ is not a singularity. On the contrary, in its Taylor expansion, we must have: $r > c$ and $r=0$ does not belong to the convergence domain. But if this expansion is truncated at any order $2n+1$, the truncated potential is defined for every $r>0$, and the point r $=0$ is a pole of order $2n+1$ for this function.

The Hamiltonian system generated by (3) admits particular solutions in the equatorial plane, defined by: $z=0$, $p_z=0$. Along these solutions Γ_h, we have (from the first integral: $H_{2n+1}=h$ and with: $p_\rho = d\rho/dt$)

$$\left(\frac{d\rho}{dt}\right)^2 = 2h - \frac{C^2}{\rho^2} + 2\sum_{p=0}^{n} \frac{c^{2p}}{r^{2p+1}} P_{2p}(0). \tag{4}$$

In order to specify the nature of these solutions Γ_h, we can define the change of time: $dt = \rho^{n+1} \cdot ds$, so that

$$\int ds = \int \left(2h \cdot \rho^{2n+2} - C^2 \cdot \rho^{2n} + 2\sum_{p=0}^{n} c^{2p} \cdot \rho^{2(n-p)+1} \cdot P_{2p}(0)\right)^{-1/2} \cdot d\rho. \tag{5}$$

The inversion of this quadrature shows that Γ_h is defined by elliptic functions $\rho(s)$ if $n=1$, or by hyperelliptic functions if $n \geqslant 2$.

To prove the non-integrability of the Hamiltonian (3), we shall first study the limit problem when $h \to -\infty$, because in this case the dominant term of the potential is homogeneous of degree $-(2n+1)$.

Let us define the small parameter: $\varepsilon = (-h)^{-1/(2n+1)}$, and the following change of scale: $(\rho, z, t) \mapsto (\varphi, \psi, \tau)$:

$$\begin{cases} \rho = \varepsilon \cdot c^{2n/(2n+1)} \cdot (P_{2n}(0))^{1/(2n+1)} \cdot \varphi \\ z = \varepsilon \cdot c^{2n/(2n+1)} \cdot (P_{2n}(0))^{1/(2n+1)} \cdot \psi \\ t = \varepsilon^{(2n+3)/2} \cdot c^{2n/(2n+1)} \cdot (P_{2n}(0))^{1/(2n+1)} \cdot (2n+1)^{-1/2} \cdot \tau \end{cases} \tag{6}$$

Then the family Γ_h, which is characterized by the first integral (4), is now defined by

$$-\frac{2n+1}{2}\left(\frac{d\varphi}{d\tau}\right)^2 = 1 + \frac{C^2}{2} c^{-4n/(2n+1)} \cdot (P_{2n}(0))^{-2/(2n+1)} \cdot \varphi^{-2} \cdot \varepsilon^{2n-1}$$
$$- \sum_{p=0}^{n} c^{-2(n-p)/(2n+1)} \cdot (P_{2p}(0))^{2(n-p)/(2n+1)}$$
$$\cdot \varphi^{-(2p+1)} \cdot \varepsilon^{2(n-p)}. \tag{7}$$

Along Γ_h, the linearized equations deduced from the Hamitonian (3) are decoupled and the normal variational equation can be written

$$\frac{d^2\xi}{dt^2} = \sum_{p=0}^{n} \frac{c^{2p}}{\rho^{2p+3}} (P''_{2p}(0) - (2p+1) P_{2p}(0)) \cdot \xi.$$

The change of scale (6) induces, for the space variable ξ, the following change:

$$\xi = \varepsilon \cdot c^{2n/(2n+1)} \cdot (P_{2n}(0))^{1/(2n+1)} \cdot \eta \tag{6'}$$

Then the normal variational equation of the truncated problem, expressed with the new variables, becomes

$$\frac{d^2\eta}{d\tau^2} = \frac{1}{2n+1} \sum_{p=0}^{n} (P''_{2p}(0) - (2p+1) P_{2p}(0)) \cdot c^{-2(n-p)/(2n+1)}$$
$$\cdot (P_{2n}(0))^{-(2p+1)/(2n+1)} \cdot \frac{\varepsilon^{2(n-p)}}{\varphi^{2p+3}} \cdot \eta. \tag{8}$$

Remark. In the normal variational Eq. (8), only two terms of each Legendre polynomial appear. Then, for the criterion of non-integrability, it will not be necessary to write these whole polynomials explicitly.

5. THE TRUNCATED PROBLEM IN THE LIMIT CASE $h = -\infty$

THEOREM 1. *The symmetric two fixed centres Problem, where the expansion of the potential has been truncated at any order* $2n+1$ ($n \in N^*$), *does not admit any first integral, meromorphic and independent of the energy and the angular momentum, in the limit case*: $h = -\infty$.

Proof. This limit problem is characterized by: $\varepsilon = 0$. Then the particular solutions $\Gamma_{-\infty}$ are defined, from (7), by

$$-\frac{2n+1}{2}\left(\frac{d\varphi}{d\tau}\right)^2 = 1 - \frac{1}{\varphi^{2n+1}}. \tag{9}$$

And the corresponding normal variational equation along $\Gamma_{-\infty}$ is now

$$\frac{d^2\eta}{d\tau^2} + \left(1 - \frac{1}{2n+1}\frac{P''_{2n}(0)}{P_{2n}(0)}\right)\frac{\eta}{\varphi^{2n+3}} = 0. \tag{10}$$

The potential function which corresponds to the particular solutions $\Gamma_{-\infty}$ is homogeneous of degree $-(2n+1)$. The comparison of (10) and (2) shows that the non-integrability coefficient of Yoshida: $\lambda_{-(2n+1)}$ is defined by

$$\lambda_{-(2n+1)} = 1 - \frac{1}{2n+1}\frac{P''_{2n}(0)}{P_{2n}(0)}. \tag{11}$$

The set $S_{-(2n+1)}$ defined in the Yoshida's criterion is

$$S_{-(2n+1)} = \,]1;\, +\infty[\, \cup\,]-2n+1;\, 0[\, \cup\,]-6n;\, -2n-2[\, \cup \cdots$$

Now, the Legendre polynomial of degree $2n$ is defined by ([W-W], page 302)

$$P_{2n}(x) = \sum_{r=1}^{n} (-1)^r \frac{(4n-2r)!}{2^{2n}\cdot r!\,(2n-r)!\,(2n-2r)!}\, x^{2(n-r)}.$$

Then

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \qquad \forall n \in N,$$

and

$$P''_{2n}(0) = (-1)^{n-1} \frac{(2n+2)!}{2^{2n}(n-1)!\,(n+1)!} \qquad \forall n \in N^*.$$

We can conclude that $P''_{2n}(0)/P_{2n}(0)$ is strictly negative for every $n \in N^*$; $\lambda_{-(2n+1)} > 1$ and then: $\lambda_{-(2n+1)} \in S_{-(2n+1)}$, $\forall n \in N^*$.

Remark. The mechanical interpretation of the limit case $h = -\infty$ (for the truncated potential) is characterized by orbits for which $r \to 0$. This limit case cannot be looked in the whole expansion of the potential, which is convergent for $r > c$. But it makes sense for the truncated system, at any order $\geqslant 3$.

6. THE TRUNCATED PROBLEM WITH FINITE VALUES OF THE ENERGY h

THEOREM 2. *The symmetric two fixed centres Problem, truncated at any order* $2n+1$ $(n \in N^*)$ *does not admit any first integral meromorphic and independent of the energy and the angular momentum, unless* $h \in]-\infty, h_0(n)[$.

Proof. We can use the same argument as used in the proof of non-integrability of the J_2-Problem [I-S]. First, we remark that, if $\varepsilon \neq 0$, the particular solutions Γ_h are defined by Eq. (7), instead of (9). But, as we saw through Eq. (5), these solutions are characterized by elliptic or hyper-elliptic functions $\rho(s)$ (or ($\varphi(\tau)$ after the scale change (6)).

If $\varepsilon = 0$ $(h = -\infty)$, the particular solutions (9) define a Riemann surface $\Gamma_{-\infty}$ and the function $\varphi(\tau)$ has $(2n+1)$ algebraic branch points. A monodromy matrix g of the normal variational equation (10) is associated to a loop on the Riemann surface and depends only on the homotopy class of this loop. The generators g_1 and g_2 chosen to construct Yoshida's criterion are associated to special loops on Γ_h [Y_1], and they verify: $\operatorname{tr} g_1 > 2$, $\operatorname{tr} g_2 > 2$ and $g_1 g_2 \neq g_2 g_1$.

When $\varepsilon \neq 0$, the particular solutions (7) define the Riemann surface Γ_h and, because the second member of (7) is analytic with respect to ε, the $(2n+1)$ branch points of $\varphi(\tau)$ are as close as desired to the branch points of the limit problem. Then, the fundamental group of the Riemann surface Γ_h is the same as in the limit problem and we can define the generators $g_1(h)$ and $g_2(h)$ of the monodromy group of the Eq. (8) by the same loops as those used in the limit problem.

Because the coefficient of η in (8) is analytic with respect to ε, along these loops $\operatorname{tr} g_1(h)$ and $\operatorname{tr} g_2(h)$ are C^0-functions of h. As we have: $\operatorname{tr} g_1(-\infty) > 2$, $\operatorname{tr} g_2(-\infty) > 2$ and $g_1(-\infty) \cdot g_2(-\infty) \neq g_2(-\infty) \cdot g_1(-\infty)$, then these conditions hold for $g_1(h)$ and $g_2(h)$, provided $|h|$ is large enough, that is to say, if $h \in]-\infty, h_0(n)[$.

7. APPLICATION TO THE VINTI PROBLEM

The Vinti Problem is a modelisation of the gravitational potential of an oblate planet, which consists in putting two equal masses on the polar axis, separated by an imaginary distance [V][B]. This Hamiltonian system is integrable because it is a particular case of the two fixed centres problem. But if we truncate the expansion of the Vinti potential, we obtain a non-integrable problem in the sense fo Ziglin :

COROLLARY. *The Vinti Problem, truncated at any order* $2n+1$ $(n \in N^*)$ *has no first integral meromorphic and independent of the energy and the angular momentum, unless* $h \in \,]-\infty, h_0'(n)[$.

Proof. The truncated Vinti Problem is defined by the Hamiltonian (3) where: $c = c'\sqrt{-1}$. Then

$$\tilde{H}_{2n+1}(\rho, z, p_\rho, p_z) = \frac{1}{2}(p_\rho^2 + p_z^2) + \frac{C^2}{2\rho^2} - \sum_{p=0}^{n} \frac{(-1)^p c'^{2p}}{r^{2p+1}} P_{2p}\left(\frac{z}{r}\right).$$

In the scaling (6) of the two fixed centres problem, if we replace c by $c'\sqrt{-1}$, we obtain particular solutions Γ_h defined by

$$-\frac{2n+1}{2}\left(\frac{d\varphi}{d\tau}\right)^2 = 1 + \frac{C^2}{2} c'^{-4n/(2n+1)} \cdot (P_{2n}(0))^{-2/(2n+1)} \cdot \varphi^{-2} \cdot \varepsilon^{2n-1}$$
$$- \sum_{p=0}^{n} (-1)^{n-p} \cdot c'^{-2(n-p)/(2n+1)} \cdot (P_{2p}(0))^{2(n-p)/(2n+1)}$$
$$\cdot\, \varphi^{-(2p+1)} \varepsilon^{2(n-p)}.$$

The normal variational equation along Γ_h is now

$$\frac{d^2\eta}{d\tau^2} = \frac{1}{2n+1} \sum_{p=0}^{n} (P''_{2p}(0) - (2p+1)\, P_{2p}(0))\, c'^{-2(n-p)/(2n+1)}$$
$$\cdot\, (P_{2n}(0))^{-(2p+1)/(2n+1)} \cdot \frac{\varepsilon^{2(n-p)}}{\varphi^{2p+3}} (-1)^{n-p} \cdot \eta.$$

This normal equation is different from (8), but in the limit problem $(\varepsilon = 0)$ the only term of the second member of this equation is obtained with $p = n$, and the normal variational equation has the same form as (10), with a coefficient of non-integrability $\lambda_{-(2n+1)}$ defined by (11). Then, by the same arguments as those used in the theorems 1 and 2, we can complete the proof of this corollary.

Remark. Another representation of the gravitational potential of an oblate planet is defined, in a first approximation, by the J_2-Problem, which is non-integrable [I-S]. Actually, this problem is exactly the Vinti Problem, truncated at the order -3 ($n=1$), provided the coefficient of oblateness J_2 verifies: $a_e^2 \cdot J_2 = c'^2$ (where a_e is the equatorial radius of the planet) [B]. But if we consider the successive approximations $J_2, J_4, J_6, \ldots$ of the potential of an oblate planet (expansion in spherical harmonics), this expansion is different from the one of the Vinti potential [B]. The identification of the two problems is possible only for the truncation at the order -3.

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